Word Games or Factorizations?

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Have you ever played that game where you can transform one word into a new word by changing one letter at a time to make intermediate words?

c o o k
_ _ _ _
_ _ _ _
_ _ _ _
l y r e

How many lines it would take to do it? Can you do it with four steps or would you need more? Can you always do it in four lines if you start with a four letter word? What would happen if I let you change two letters at once instead of one?

In factorization theory, there is an idea called the catenary degree. It measures how far apart two factorizations are from each other. Here is a definition of the catenary degree from [1].

> The *catenary degree* of an element, a, is the smallest integer c such that for any two finite factorizations of a there is a chain of factorizations which connects them and satisfies the property that the distance between two consecutive links is no more than c.

This time our "words" are factorizations and our "letters" are the irreducibles present in the factorizations. Two of the factorizations of the element 36 in $\mathbb{Z}[\sqrt{-5}]$ are $2^2 \cdot 3^2$ and $(1+\sqrt{-5})^2(1-\sqrt{-5})^2$. Both factorizations consist of four irreducibles. You can connect these factorizations if you change two irreducibles at a time as the catenary degree of 36 in $\mathbb{Z}[\sqrt{-5}]$ happens to be 2. Make sure that each time you do this, however, you keep the final product as 36 when you multiply it out.

$$
\begin{array}{ccccccccc}
(1+\sqrt{-5}) & \cdot & (1+\sqrt{-5}) & \cdot & (1-\sqrt{-5}) & \cdot & (1-\sqrt{-5}) & = & 36 \\
\_ & \cdot & \_ & \cdot & \_ & \cdot & \_ & = & 36 \\
\underline{2} & \cdot & \underline{2} & \cdot & \underline{3} & \cdot & \underline{3} & = & 36
\end{array}
$$

We will explore this notion and build a graph that will let us find what the catenary degree of an element is.

References

[1] Chapman, S. T., García Sánchez, P.A., Llena, D., Ponomarenko, V., Rosales, J.C.: The catenary and tame degree in finitely generated commutative cancellative monoids. Manuscripta Math. 120 no. 3, 253-264 (2006)

(See next page for answers.)

$$\begin{array}{cccc} \mathrm{c} & \mathrm{o} & \mathrm{o} & \mathrm{k} \\ \mathrm{c} & \mathrm{o} & \mathrm{r} & \mathrm{k} \\ \mathrm{c} & \mathrm{o} & \mathrm{r} & \mathrm{e} \\ \mathrm{l} & \mathrm{o} & \mathrm{r} & \mathrm{e} \\ \mathrm{l} & \mathrm{y} & \mathrm{r} & \mathrm{e} \end{array}$$

$$\begin{array}{ccccccc} (1+\sqrt{-5}) & \cdot & (1+\sqrt{-5}) & \cdot & (1-\sqrt{-5}) & \cdot & (1-\sqrt{-5}) \\ 2 & \cdot & (1+\sqrt{-5}) & \cdot & 3 & \cdot & (1-\sqrt{-5}) \\ 2 & \cdot & 2 & \cdot & 3 & \cdot & 3 \end{array}$$